2.2 - Separable Equations

Definition: A **differential equation** (DE) is an equation containing the derivatives of one or more unknown functions (dependent variables) with respect to one or more independent variables.

Examples: $\frac{dy}{dx} + 5y = e^x$, $y'' - y' + 6y = 0$, and $\frac{dx}{dt} + \frac{dy}{dt} = 2x + y$ are ordinary differential equations (ODEs); $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is an example of a partial differential equation (PDE). In this course, we will only work with ODEs.

Definition: The **order** of a DE is the order of the highest derivative in the equation.

Definition: A first-order **separable** DE has the form $\frac{dy}{dx} = g(x)h(y)$.

Example: Solve the given differential equation by separation of variables.

$dy - (y-1)^2 dx = 0$

Note: This is a nonlinear DE because the equation contains a nonlinear function of y (the dependent variable).

Definition: Any function ϕ, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an nth-order ODE reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

Example: $\sin 3x \; dx + 2y \cos^3 3x \; dy = 0$

Example: $\dfrac{dN}{dt} + N = Ne^{t+2}$

Note: This is a linear DE because all instances of N (the dependent variable) are linear.

Definition: Information such as $y(x_0) = y_0$ is called an **initial condition**. A DE, together with an initial condition is called an **initial-value problem** (IVP).

Example: Find an explicit solution of the given initial-value problem. Determine the exact interval I of definition by analytical methods. Use a graphing utility to plot the graph of the function.

$$(2y-2)\frac{dy}{dx} = 3x^2 + 4x + 2, \qquad y(1) = -2$$

Theorem: Existence of a Unique Solution

Consider the first-order initial-value problem

Solve $\dfrac{dy}{dx} = f(x, y)$, subject to $y(x_0) = y_0$

Let R be a rectangular region in the xy-plane defined by $a \leq x \leq b,\ c \leq y \leq d$ that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ are continuous on R, then there exists some interval $I_0\ :\ (x_0 - h, x_0 + h),\ h > 0$, contained in $[a,\ b]$, and a unique function $y(x)$, defined on I_0, that is a solution of the initial-value problem.

We will consider higher-order DEs later in the course.

Example: Find an explicit solution of the given initial-value problem. Give the solution using a definite integral.

$$\frac{dy}{dx} = y^2 \sin x^2, \qquad y(-2) = \frac{1}{3}$$